§ Stability and Locally linear system

20-11-18

§ Case 3: Repeated eigenvalue r.

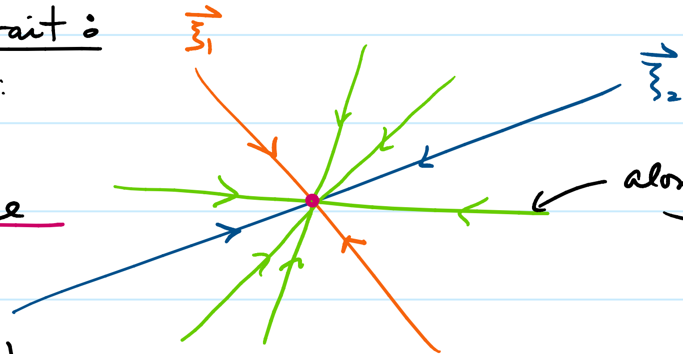
Case 3a) geometric multiple of $r = 2$

with two eigenvectors $\vec{\xi}_1, \vec{\xi}_2$.

General solution: $\vec{y}(t) = e^{rt}(c_1\vec{\xi}_1 + c_2\vec{\xi}_2)$

phase portrait: for $r > 0$:

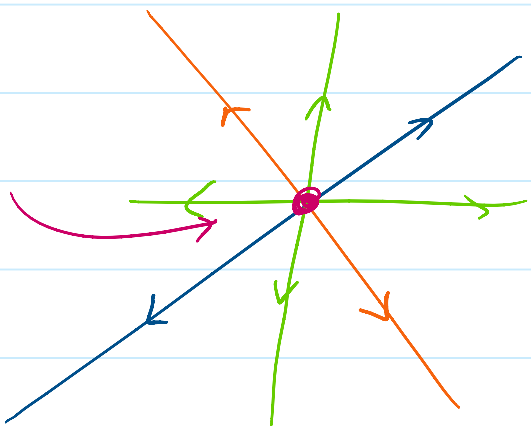
Proper node



along $c_1\vec{\xi}_1 + c_2\vec{\xi}_2$.

phase portrait for $r > 0$:

Proper node

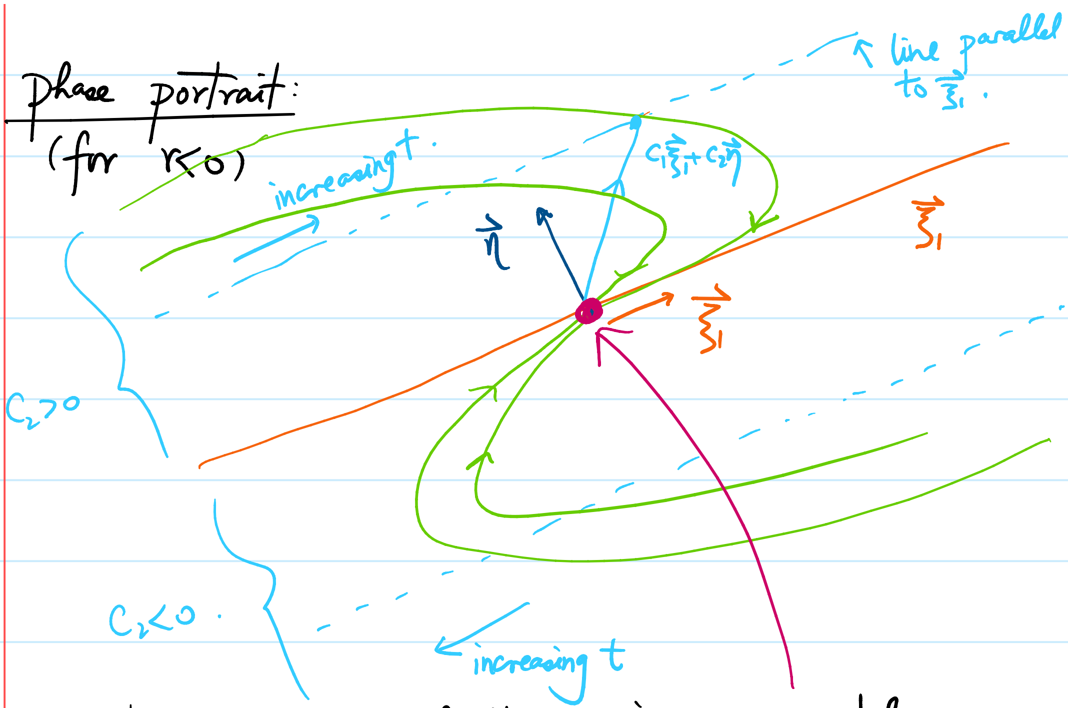


Case 3b) $\vec{\xi}$ eigenvector and $\vec{\eta}$ generalized eigenvector

general solution: $\vec{y}(t) = e^{rt}(c_1\vec{\xi}_1 + c_2(\vec{\eta} + t\vec{\xi}_1))$

$= e^{rt}((c_1 + tc_2)\vec{\xi}_1 + c_2\vec{\eta})$.

Phase portrait:
(for $r<0$)



In this case we call it an improper nodal.

- Notice that the case for $r>0$ is obtained by reversing the direction of trajectory.

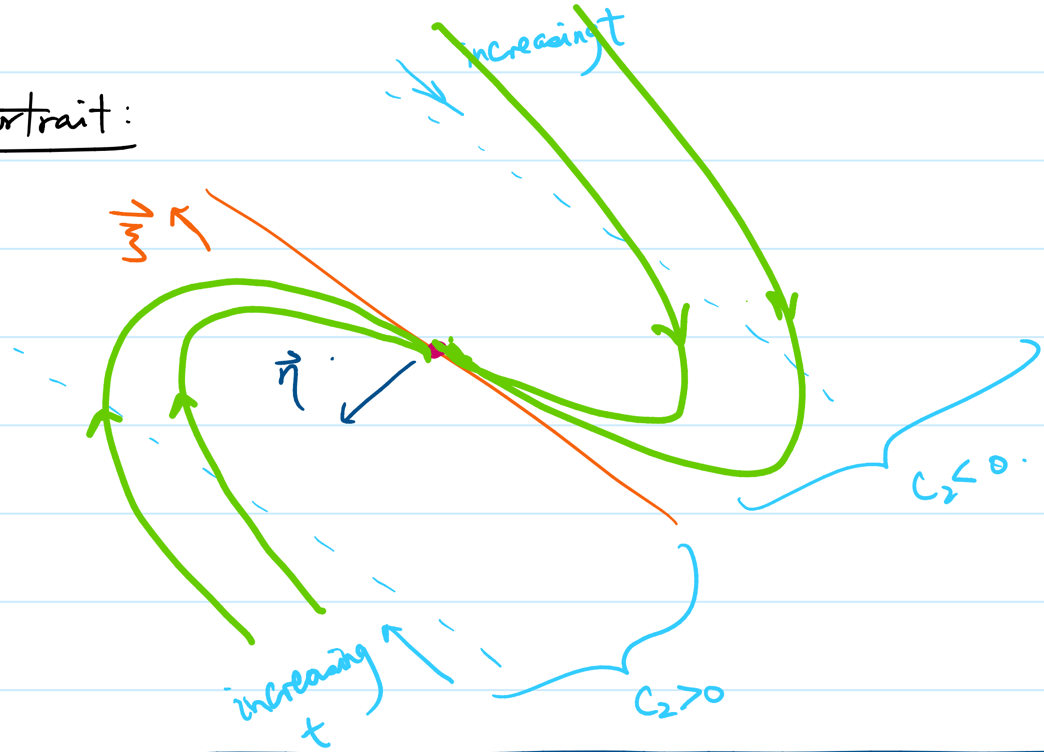
Eg.3: $A = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix}$ with $r=-3$ repeated eigenvalue

and $\vec{\xi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ eigenvector

$\vec{\eta} = \frac{-1}{8}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ generalized eigenvector.

General sol: $\vec{y}(t) = e^{-3t}\left((c_1 + tc_2)\vec{\xi} + c_2\vec{\eta} \right)$.

phase portrait:



Def: • We consider $\vec{y}'(t) = f(\vec{y}(t))$ on $(-a,+\infty)=I$.
with $y_* \in \mathbb{R}^n$ be a point s.t. $\boxed{f(y_*)=0}$.
we call it critical point of the system.
and hence $\vec{y}(t) \equiv y_*$ is the solution through y_*.

1) $\vec{y}_*$ is called a stable critical point if
for any $\varepsilon>0$, $\exists \delta>0$ (depending on y_* & ε)
s.t. for any solution $\vec{y}(t)$ satisfying
$$|\vec{y}(0)-y_*|<\delta \Longrightarrow |\vec{y}(t)-y_*|<\varepsilon \text{ for } t\geq 0$$

2) $\vec{y}_*$ is called unstable if it is NOT stable.

3) $\vec{y}_*$ is called asymptotically stable if it is stable, and $\exists \delta_0>0$ (depending on y_*) s.t.
$$|\vec{y}(0)-y_*|<\delta_0 \Longrightarrow \lim_{t\to\infty}\vec{y}(t)=\vec{y}_*.$$

Summary:

Eigenvalues	Type of $\vec{y}_* = \vec{0}$	Stability
$r_1 < r_2 < 0$	Node	Asym. stable.
$r_1 < 0 < r_2$	saddle	unstable.
$0 < r_1 < r_2$	Node	unstable
$r_1 = r_2 < 0$	Proper / Improper node	Asym. stable
$r_1 = r_2 > 0$	Proper / Improper node	Unstable
$\lambda = \alpha + i\mu,\ \alpha < 0$	Spiral	Asym. stable
$\lambda = \alpha + i\mu,\ \alpha > 0$	Spiral	unstable.
$\lambda = i\mu$	Center	stable but <u>NOT</u> asym. stab.

(Eg) (Case of non-isolated critical point)

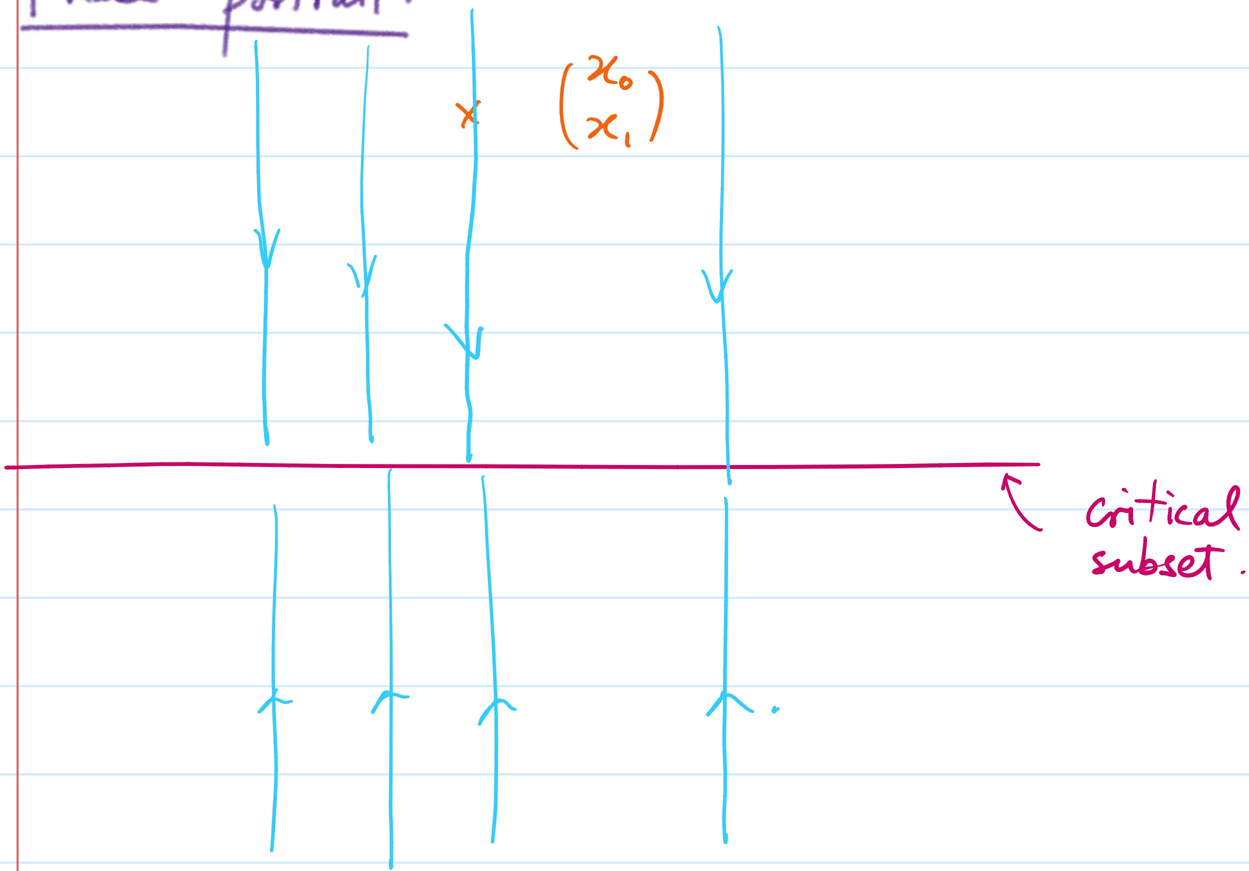
- Let $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ and consider $\vec{y}' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \vec{y}$.

$$\ker(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

- For any initial value $\vec{y}(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we know the solution is of the form:

$$y_1(t) \equiv x_1, \qquad y_2(t) = x_2 e^{-t}$$

Phase portrait:



§ Locally Linear system:

Perturbation & eigenvalues:

E.g 1: • $\vec{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{y} \Rightarrow$ eigenvalues $= \pm i$

and it is a stable center.

• if we add $A_\varepsilon = \begin{pmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{pmatrix}$

eigenvalue $= \varepsilon \pm i$

either: A. stable spiral $(\varepsilon < 0)$, or Unstable spiral $(\varepsilon > 0)$

Eg 2: $A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, $A_{\delta,\varepsilon} = \begin{pmatrix} -1+\delta & 1 \\ -\varepsilon & -1 \end{pmatrix}$

repeated

$\delta = 0, \varepsilon = 0$: eigenvalues are -1 : A. stable improper nodal

$\delta = 0, \varepsilon > 0$: eigenvalues are $-1 \pm \sqrt{\varepsilon}\, i$: A. stable spiral

$\delta = 0, \varepsilon < 0$: " are $-1 \pm \sqrt{\varepsilon}$: A. stable nodal.

$\delta \neq 0, \varepsilon \neq 0$: eigenvalues are real and distinct : A. stable nodal

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Def: We consider 2×2 system of the form

$$\vec{y}\,' = f(\vec{y}) \quad \text{------} \quad (*)$$

$\vec{y}_*$ is called isolated critical point if there is a neighborhood $\mathcal{U}$ of $\vec{y}_*$ s.t. $\vec{y}_*$ is the only zero of f in $\mathcal{U}$

idea: • If $\vec{y}_*$ is an isolated critical pt, f is differentiable

$$\Longrightarrow \quad f(\vec{y}) - f(\vec{y}_*) = Df(\vec{y}_*)(\vec{y} - \vec{y}_*) + \vec{g}(\vec{y}) \text{ with}$$

$$\frac{\| g(\vec{y}) \|}{\| \vec{y} - \vec{y}_* \|} \longrightarrow 0 \quad \text{as} \quad \| \vec{y} - \vec{y}_* \| \longrightarrow 0.$$

$C^1 \Rightarrow$ differentiable

Jacobian matrix: $Df(\vec{y}_*) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}\right|_{\vec{y} = \vec{y}_*}$

☆: Solution of (*) near $\vec{y}_*$ is well approximated by the Linear system $\vec{z}\,' = Df(\vec{y}_*)\cdot(\vec{z})$ for $\vec{z} = \vec{y} - \vec{y}_*$.

Def: $\vec{y}_*$ is an isolated critical point of (*), we say (*) is locally linear near $\vec{y}_*$ if

i) $f(\vec{y})$ differentiable at $\vec{y}_*$

ii) $Df(\vec{y}_*)$ invertible.

E.g.

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -w^2 \sin y_1 - \gamma y_2 \end{cases} \Rightarrow f(y_1, y_2) = \begin{pmatrix} \underbrace{y_2}_{= f_1(y_1,y_2)} \\ \underbrace{-w^2 \sin y_1 - \gamma y_2}_{= f_2(y_1,y_2)} \end{pmatrix}$$

$$(Df)(\vec{y}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 \cos y_1 & -\gamma \end{pmatrix}$$

and since $\dfrac{\partial f_i}{\partial y_j}$ are all continuous

$\Longrightarrow$ $f(\vec{y})$ is differentable.

$f(\vec{y}) = 0 \Longrightarrow -w^2 \sin y_1 = 0,\ y_2 = 0 \Rightarrow y_1 = k\pi,\ y_2 = 0$

$\det(Df(\vec{y})) = 0 \Rightarrow w^2 \cos y_1 = 0 \Longrightarrow y_1 = \dfrac{k\pi}{2}.$

$\therefore$ the system is locally linear near every critical point $y_1 = k\pi$, $y_2 = 0$

$k \in \mathbb{Z}$.

Thm: • Let $\vec{0}$ be an isolated critical point of $(*)$, and $(*)$ is locally linear near $\vec{0}$, we let $A = Df(\vec{0})$ be the Jacobian matrix, let r_1, r_2 be eigenvalues of A.

• Besides the case a) $r_1 = i\mu$, $r_2 = -i\mu$

b) $r_1 = r_2 \in \mathbb{R}$.

the type & stability of $(*)$ near $\vec{0}$, and the linear system $\vec{y}' = A\vec{y}$ are the same.

i.e.

r_1, r_2	type	stability
$0 < r_1 < r_2$	Node	unstable
$r_1 < r_2 < 0$	Node	A. stable
$r_1 < 0 < r_2$	saddle	unstable
$r_1, r_2 = \lambda \pm i\mu$, $\lambda > 0$	Spiral	unstable
$r_1, r_2 = \lambda \pm i\mu$, $\lambda < 0$	spiral	A. stable

- In the other case, we only have partial information.

r_1, r_2	Type	stability
$r_1 = r_2 > 0$	Node or Spiral	unstable
$r_1 = r_2 < 0$	Node or Spiral	A-stable.
$r_1, r_2 = \pm i\mu$	Center of Spiral	undetermined.

Rk: • the above cases

a) $r_1, r_2 = \pm i\mu$

b) $r_1 = r_2$

correspond to the possibility that we obtain in earlier example about perturbation & eigenvalues.

- This effect comes from the small term $g(\vec{y})$ in the approximation of locally Linearity. Even $g(\vec{y})$ is small, it may cause a qualitative change in behaviour of eigenvalues.

Rk: The proof is beyond scope of this course.